

# On a Probabilistic Interpretation of Relativistic Quantum Mechanics

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A probabilistic interpretation of one-particle relativistic quantum mechanics is proposed. Quantum Action Principle formulated earlier is used for to make the dynamics of the Minkowsky time variable of a particle to be classical. After that, quantum dynamics of a particle in the 3D space obtains the ordinary probabilistic interpretation. In addition, the classical dynamics of the Minkowsky time variable may serve as a tool for "observation" of the quantum dynamics of a particle. A relativistic analog of the hydrogen atom energy spectrum is obtained.

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## I. INTRODUCTION

The subject of the present work is the problem of a probabilistic interpretation of relativistic quantum mechanics (RQM) based on the Klein-Gordon (KG) wave equation for a particle in an external electromagnetic field (see, for example, [1]),

$$\hat{F}\psi \equiv \left[ \left( \frac{\hbar}{i} \nabla_\mu - eA_\mu \right) \left( \frac{\hbar}{i} \nabla^\mu - eA^\mu \right) - m^2 c^2 \right] \psi = 0. \quad (1)$$

In the present paper we develop an approach that may be called "refreshed" dynamics in contrast to the term "frozen" dynamics, which is used in quantum theory of the universe based on the Wheeler-De Witt wave equation [2]. The "refreshment" of the one-particle quantum dynamics will be performed by use of a quantum action principle (QAP) [3] which was applied first to the problem of probabilistic interpretation of RQM in [4]. In QAP the KG equation (1) is replaced by the Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial s} = \hat{F}\psi, \quad (2)$$

where  $s \in [0, S]$  is an internal time parameter of a particle. The time interval length  $S$  is an additional dynamical variable of a particle which is not observable. It has to be excluded by use of a condition of the stationarity of a quantum action with respect to small variations of  $S$ . It is this condition of the stationarity that we call QAP. The quantum action is defined as the real phase of a transition amplitude of a particle in a real experiment. In the present work QAP is supplied by some more conditions of the stationarity, which makes the dynamics of the Minkowsky time dynamical variable  $x^0 = ct$  to be classical. In addition to the "ordinary" probabilistic interpretation of RQM, the classical dynamics of the Minkowsky time gives us a tool to "observe" the quantum dynamics

of a particle "from the inside", and develops its alternative interpretation. The latter will be useful in quantum cosmology.

## II. QUANTUM ACTION PRINCIPLE IN RELATIVISTIC MECHANICS

Schrödinger equation (2) arises in the result of the standard quantization procedure applied to the classical action with an invariant time parameter [5],

$$I = \int_0^S ds \left\{ p_\mu \dot{x}^\mu - [(p_\mu - eA_\mu)(p^\mu - eA^\mu) - m^2 c^2] \right\}. \quad (3)$$

The upper limit of the integration has to be defined from the condition of the stationarity of (3) after substitution into it a solution of classical equations of motion. It is proportional to a proper time. For a free moving particle it equals

$$S = \frac{\sqrt{(x_1 - x_0)^2}}{2mc}, \quad (4)$$

where  $x_{0,1}^\mu$  are the end points of a world line of a particle. The main idea of the work [4] was that to "delay" the condition of stationarity up to the quantum level. In the present work we slightly "improve" the action (3) enlarging the set of dynamical variables as follows:

$$\begin{aligned} \tilde{I} = \int_0^S ds \left\{ p_\mu \dot{x}^\mu - [d^2 - (p_i - eA_i)^2 - m^2 c^2] \right. \\ \left. + \lambda [d - (p_0 - eA_0)] \right\}. \end{aligned} \quad (5)$$

Two additional variables,  $d$  and  $\lambda$ , may be excluded at the classical level, and in the result the action (5) comes back to the form (3). But, once again, we delay this exclusion up to the quantum level. A result of this delay will be a classical character of the dynamics of the Minkowsky time parameter  $x^0(s)$ , which, in turn, "refreshes" the dynamics of particle.

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Let us write the Schrödinger equation corresponding to the action (5),

$$i\hbar \frac{\partial \psi}{\partial s} = \left[ d^2(s) - \lambda(s) \left( d(s) + i\hbar \frac{\partial}{\partial x^0} - k \frac{e^2}{cr} \right) - m^2 c^2 + \hbar^2 \Delta \right] \psi, \quad (6)$$

where for simplicity we consider an electron movement in the Coulomb field of a hydrogen atom. In this case the dependence of a solution  $\psi$  on the variable  $x^0$  may be considered separately. One write  $\psi(s, x^\mu) = \psi_0(s, x^0) \varphi(s, x^i)$ , where

$$\psi_0(s, x^0) = \exp \chi(s, x^0). \quad (7)$$

Here  $\chi(s, x^0)$  is a complex phase for which a quadratic representation

$$\chi(s, x^0) = \chi_0(s) + \chi_1(s) x^0 + \frac{1}{2} \chi_2(s) (x^0)^2 \quad (8)$$

is sufficient. From the equation (6) we obtain a set of ordinary differential equations for coefficients  $\chi_{0,1,2}$ ,

$$\begin{aligned} i\hbar \dot{\chi}_0 &= d^2 - \lambda d - m^2 c^2 + i\hbar \lambda \chi_1, \\ i\hbar \dot{\chi}_1 &= i\hbar \lambda \chi_2, \\ \dot{\chi}_2 &= 0. \end{aligned} \quad (9)$$

We consider the quadratic representation (8) according to the Gauss form of the initial wave function:

$$\psi_0(0, x^0) = A \exp \left( -\frac{(x^0)^2}{4\sigma^2} \right). \quad (10)$$

A solution of the set (9) corresponding to the initial wave function (10) is

$$\begin{aligned} \chi_2(s) &= -\frac{1}{2\sigma^2}, \\ \chi_1(s) &= \frac{1}{2\sigma^2} \int_0^s ds' \lambda(s'), \\ \chi_0(s) &= \frac{1}{i\hbar} \int_0^s ds' (d^2 - \lambda d - m^2 c^2) \\ &\quad - \frac{1}{2\sigma^2} \int_0^s ds' \lambda(s') \int_0^{s'} ds'' \lambda(s''). \end{aligned} \quad (11)$$

To complete the formulation of QAP one needs a remaining part of the phase of a full wave function  $\psi(s, x^\mu)$ . The remaining part  $\varphi(s, x^i)$  of the wave function obeys the Schrödinger equation,

$$i\hbar \frac{\partial \varphi}{\partial s} = \hat{H}(\lambda(s)) \varphi \equiv - \left[ -\hbar^2 \Delta - k \lambda(s) \frac{e^2}{cr} \right] \varphi. \quad (12)$$

In order to examine QAP before its application for the problem of the probabilistic interpretation let us finish the section by consideration of a stationary solution of the equation (12) for a hydrogen atom. Supposing  $\lambda = \text{const}$ , we consider the stationary Schrödinger equation,

$$\hat{H} \varphi = \epsilon \varphi. \quad (13)$$

The equation (13) has the set of bound eigenstates with the eigenvalues

$$\epsilon_n = -\frac{\lambda^2}{2mc^2} E_n, \quad E_n \equiv -\frac{\hbar R}{n^2}, \quad (14)$$

where  $R$  is the Rydberg constant, and  $n = 1, 2, \dots$  is the principal quantum number, and corresponding eigenfunctions are

$$\varphi_n \left( \frac{2mc}{\lambda} x^i \right), \quad (15)$$

where  $\varphi_n(x^i)$  are ordinary non-relativistic stationary wave functions of an electron in the hydrogen atom. In this stationary case the remaining part of the full phase equals  $(1/i\hbar) \epsilon_n s$ , and the real part of the phase is

$$\left( d^2 - \lambda d - m^2 c^2 - \frac{\lambda^2}{2mc^2} E_n \right) s. \quad (16)$$

One can see from the representation (8) and the solution (11) for its coefficients that

$$\begin{aligned} \psi_0(s, x^0) &= A \exp \left[ -\frac{(x^0 - \lambda s)^2}{4\sigma^2} \right. \\ &\quad \left. - \frac{i}{\hbar} \left( d^2 - \lambda d - m^2 c^2 - \frac{\lambda^2}{2mc^2} E_n \right) s \right]. \end{aligned} \quad (17)$$

It means that  $\lambda S$  equals to the end value of the Minkowsky time coordinate  $x_1^0$ . The quantum action defined as the real phase (16) with account of this additional condition is

$$\begin{aligned} \Lambda &= \left( d^2 - \lambda d - m^2 c^2 - \frac{\lambda^2}{2mc^2} E_n \right) S \\ &\quad + \varkappa (\lambda S - x_1^0), \end{aligned} \quad (18)$$

where  $\varkappa$  is a corresponding Lagrangian multiplier. We formulate QAP as a set of conditions of the stationarity of the action (18) with respect to small variations of the parameters  $d, \lambda, S, \varkappa$ . Their stationary values are

$$\begin{aligned} \lambda &= 2d = 2mc / \sqrt{1 + \frac{2E_n}{mc^2}}, \quad S = \frac{x_1^0}{\lambda}, \\ \varkappa c &= \sqrt{m^2 c^4 + 2mc^2 E_n}. \end{aligned} \quad (19)$$

Compare the second equation with the classical result in the Eq. (4). It is the last quantity in (19) that equals to the energy of an electron in the hydrogen atom in our theory (as the coefficient in front of  $t = x^0/c$ ). This result

coincides with the prediction of the Dirac equation with accuracy up to the square of the fine structure constant,  $\alpha^2$ . In the precise energy spectrum the square of the principal quantum number  $n^2$  must be replaced by the following quantity [6]:

$$\begin{aligned} n^{*2} &= p^2 + 2p\sqrt{k^2 - \alpha^2} + k^2, \\ p &= 0, 1, \dots, |k| = 1, 2, \dots \end{aligned} \quad (20)$$

The origin of the difference is the absence of spin in our theory.

### III. PROBABILISTIC INTERPRETATION OF RELATIVISTIC QUANTUM MECHANICS

Let us turn to dynamics. Let  $\lambda(s)$  be an arbitrary function of the internal time, and  $\hat{U}_S[\lambda(s)]$  be the evolution operator for the Schrödinger equation (12) on the time interval  $[0, S]$ . For a pair of normalized states  $|\varphi_{in}\rangle$  and  $|\varphi_{out}\rangle$  one can introduce a transition amplitude,

$$K_S[\lambda(s)] \equiv \langle \varphi_{out} | \hat{U}_S[\lambda(s)] | \varphi_{in} \rangle. \quad (21)$$

In the work [4] we defined a quantum action as the real phase of the transition amplitude for a concrete experiment. Here the transition amplitude (21) written in the exponential form,

$$K_S[\lambda(s)] = \exp \left\{ \frac{1}{i\hbar} I[\lambda(s)] + Q[\lambda(s)] \right\}, \quad (22)$$

defines a part of a full quantum action that, instead of the Eq.(18), equals to

$$\Lambda = - \int_0^S ds \left[ \frac{\lambda^2}{4} + m^2 c^2 \right] + \varkappa \left( \int_0^S ds \lambda - x_1^0 \right) + I[\lambda(s)]. \quad (23)$$

Let  $\tilde{\lambda}(s)$  be a stationary value of the function  $\lambda(s)$  on the time interval  $[0, \tilde{S}]$  with a stationary length  $\tilde{S}$ . Then the transition amplitude  $K_{\tilde{S}}[\tilde{\lambda}(s)]$  becomes a complex amplitude of a probability, namely, the probability of the quantum transition  $|\varphi_{in}\rangle \rightarrow |\varphi_{out}\rangle$  at the moment  $x_1^0$  of the Minkowsky time equals

$$P_{x_1^0} = \left| K_{\tilde{S}}[\tilde{\lambda}(s)] \right|^2. \quad (24)$$

This interpretation does not depend on the dynamics of the Minkowsky time variable  $x^0(s)$  which in fact is classical one. The latter follows from the Gauss form dependence on  $x^0$  of the wave function (17). In the limit  $\sigma \rightarrow 0$  it becomes proportional to the corresponding  $\delta$ -function. But the classical dynamics of the time variable  $x^0(s)$  may be considered as a tool for observation of quantum dynamics of a particle in the 3D space. In a general case, as a consequence of the classical dynamics of  $x^0(s)$ , we have

$$\tilde{\lambda}(s) = \frac{dx^0(s)}{ds}. \quad (25)$$

For a fixed initial state  $|\varphi_{in}\rangle$  and an arbitrary final state  $|\varphi_{out}\rangle$ , QAP gives a certain stationary function  $\tilde{\lambda}(s)$  and corresponding evolution of the internal time,

$$s(x^0) = \int_0^{x^0} \frac{dx^0}{\tilde{\lambda}(s(x^0))}. \quad (26)$$

We suppose that the equation (26) gives a one-to one correspondence between the quantum transition  $|\varphi_{in}\rangle \rightarrow |\varphi_{out}\rangle$  and the evolution of the internal time. Let us imagine for a moment a clock connected with a quantum electron. The movement of the clock reflects the quantum dynamics of the electron. Of course, this speculation is unreal. But in quantum cosmology just this situation is realized: all observers with devices are located in a quantum universe. We need in this case an infinite set of classical degrees of freedom. Their relative dynamics can be made observable.

### IV. CONCLUSIONS

Therefore, QAP gives a possibility of the ordinary probabilistic interpretation of relativistic quantum mechanics, under the condition that the dynamics of the Minkowsky time variable of particle is classical one. The latter may be achieved by corresponding modification of QAP. This approach may be useful in quantum cosmology as a method of "refreshment" of quantum dynamics of the universe.

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